Extending Special Relativity to Account for Changes in Mass of an Observer

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Abstract

Special Relativity often makes use of the abstract concept of a Reference Frame, which is made specific by describing a system of physical rods and clocks - the measuring devices of space-time. In this paper, we replace that abstract concept with the more concrete concept of an observing particle, which may carry energy, momentum, and mass. We discover that the Lorentz Transformation assumes that the observing particle, and the particle being observed, both have the same mass. The Extended Lorentz Transformation is introduced, in order to remove this assumption. The result is that the local coordinates of particles are not only related by a space-time rotation, but also a uniform space-time scale. The presence of this space-time scale factor α allows us to directly identify the abstract rods and clocks with the physical wavelength and frequency attributes of the particle. The cases where this scale factor α will produce non-trivial effects are when the mass of the observing particle changes, as occurs when the potential energy of a particle changes. Scenarios involving gravitational potential energy may be analyzed in the context of Special Relativity without requiring an equivalent accelerated reference frame. Because the Extended Lorentz Transformation is not normalized relative to the mass of the particle, it can be used to relate space-time coordinates of massless particles.

1 Identifying the Implict Mass Constraint in the Lorentz Transformation

The Lorentz Transformation, as generally expressed, is parameterized in terms of velocity. The standard parameters for this transformation are γ and β, which are defined in terms of the relative velocity v.

$$\beta = \frac{v}{c} \tag{1}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2}$$

Given these parameters, for v along the x axis, a general Lorentz Transformation takes the standard form (suppressing the trivial y and z axes):

$$ct' = \gamma' ct - \gamma' \beta' x \tag{3}$$
$$x' = \gamma' x - \gamma' \beta' ct \tag{4}$$

The primed coordinates represent the local coordinates of the moving system. We also adopt the convention of adding a prime to the velocity based parameters γ and β in order to convey the fact that they represent the relative velocity of the system with primed local coordinates. This transform can be characterized as a space-time rotation.

If we take the reverse transform we can express the coordinates of the observer in terms of the velocity based parameters of the moving frame, as well as the local coordinates of that frame.

$$ct = \gamma' ct' + \gamma' \beta' x' \tag{5}$$
$$x = \gamma' x' + \gamma' \beta' ct' \tag{6}$$

We can use this form to relate two moving frames, one noted with a prime, and the other noted with a double prime, which are observed by the common observer at rest.

$$\gamma'' ct'' + \gamma'' \beta'' x'' = \gamma' ct' + \gamma' \beta' x' \tag{7}$$
$$\gamma'' x'' + \gamma'' \beta'' ct'' = \gamma' x' + \gamma' \beta' ct' \tag{8}$$

While this isn't expressed in the standard form, we can think of this relationship as the transformation between the single prime and double prime coordinate systems, using velocity parameters determined by the shared observer.

Instead of thinking of these reference frames abstractly, we can think of them concretely as two moving particles under observation from the common observer. We choose to express the velocity based parameters γ and β in terms of the energy E and momentum p of a particle with mass m.

$$\gamma = \frac{E}{mc^2} \tag{9}$$
$$\gamma\beta = \frac{p}{mc} \tag{10}$$

The relationship between the local coordinates of the two particles can now be given in terms of the respective energy, momentum, and mass of those particles, as determined by the common observer.

$$\frac{1}{m''c}\left(E''t'' + p''x''\right) = \frac{1}{m'c}\left(E't' + p'x'\right) \tag{11}$$
$$\frac{1}{m''c}\left(\frac{E''}{c}x'' + p''ct''\right) = \frac{1}{m'c}\left(\frac{E'}{c}x' + p'ct'\right) \tag{12}$$

Were the two particles to share the same mass, the normalizing factor would cancel on both sides, and the relationship between the local coordinates of the particles would very simply by expressed in terms of their energy and momentum. Alternatively, we might interpret this as the relationship between local coordinates given in terms of *normalized* energy and momentum - namely the energy and momentum of particles which each have a mass given by $mc = 1$. From this we can see that *the Lorentz Transformation carries an implicit constraint assuming the mass of the relative observers to be identical.*

2 Extending the Lorentz Transformation

If we wish to remove this implicit constraint, it will require removing the normalization by mc.

$$E''t'' + p''x'' = E't' + p'x' \tag{13}$$

$$\frac{E''}{c}x'' + p''ct'' = \frac{E'}{c}x' + p'ct' \tag{14}$$

Remember, the form we are working with here is technically the reverse transformation. We want to get back into standard form. We begin by placing the double prime particle into the same state as the shared observer, namely we let p'' go to zero. The immediate consequence of this is that $E'' = m''c^2$.

$$(m''c)\,ct'' = E't' + p'x' \tag{15}$$

$$(m''c)\,x'' = \frac{E'}{c}x' + p'ct' \tag{16}$$

Now, rather than claiming that we are expressing a relationship between two concrete particles viewed by an abstract reference frame, we allow the concrete double prime particle to take on the role of the physical observer. This is expressed by removing the double primes altogether. The particle under observation is the single prime particle.

We invert the transform, so that the local coordinates of the moving particle (single prime) are expressed in terms of the observing particle.

$$ct' = \frac{mc}{(m'c)^2}\left(E't - p'x\right) \tag{17}$$

$$x' = \frac{mc}{(m'c)^2}\left(\frac{E'}{c}x - p'ct\right) \tag{18}$$

Finally, we return to expressing the full transformation, across all axes, in terms of velocity based parameters γ and β.

$$ct' = \alpha\left(\gamma' ct - \gamma'\beta' x\right) \tag{19}$$
$$x' = \alpha\left(\gamma' x - \gamma'\beta' ct\right) \tag{20}$$
$$y' = \alpha y \tag{21}$$
$$z' = \alpha z \tag{22}$$

Where the parameter α is defined as:

$$\alpha = \frac{m}{m'} \tag{23}$$

We see that the final consequence of removing the implicit "similar mass" constraint from the Lorentz Transformation is that it picks up an overall uniform space-time scale α, which is the ratio of the observing mass to the observed mass. We define the **Extended Lorentz Transformation** to be a space-time rotation as well as a uniform space-time scale, where the scale α is determined by the ratio of masses of the respective observers.

3 Relationship between α and Einstein's uniform space-time scale

In Einstein's original 1905 treatise on Special Relativity, he outlines a derivation of the Lorentz Transformation which incorporates a thorough and particular consideration for a possible uniform space-time scale factor - a factor completely equivalent to the α introduced above. This original derivation allowed for, and possibly even suggested, the existence of such a scale factor, which Einstein did not fail to recognize. This factor is so important that it occupies more than half of the section devoted to the derivation of the transform, before finally concluding that the value of the scale factor must be unity.

We now walk through these considerations, and that final conclusion.

The derivation starts by defining the length of a moving object in terms of the round trip time it takes light to traverse the bounding edges of that object. Using this definition, Einstein determines differential relationships between the coordinates of a moving and non-moving frame. To begin with, he recognizes that these relationships admit a transform which includes a space-time scale, but determines that this scale cannot be directly dependent on the space or time coordinates of an inertial frame. It may only be dependent on the *magnitude* of the velocity of the moving frame, and so he denotes the scale factor as $\phi(v)$.

Next, he develops the entire Lorentz Transformation, as we currently know it, yet including an overall space-time scale, as has been done here in equations (19) through (22).

He shows that such a Transformation must satisfy the two constraints which were imposed at the outset, namely:

- It satisfies the Principle of Relativity as laid out in that paper, and

- It satisfies the condition that the speed of light is constant for each reference frame.

He then determines that the scale factor $\phi(-v)$ of an associated Inverse Lorentz Transform must be the multiplicative inverse of the Forward Lorentz scale factor. In other words:

$$\phi(v) = \frac{1}{\phi(-v)} \tag{24}$$

There is some brief discussion, at this point, about the significance of $\phi(v)$ in terms of how this scale will influence lengths that are measured along axes that are perpendicular to the direction of motion. The expression he derives at this point is very much equivalent to the expressions (21) and (22) determined in this paper.

At the final step of the derivation, Einstein could have concluded that $\phi(v)$ might possibly be interpreted as a unit conversion between the different reference frames. Instead, there was an unspoken, or implicit, assumption that the units of measure between the coordinate systems should be the same. That assumption takes explicit form, when, by consequence of the fact that the perpendicular axes should not change length, he states that the overall conversion, either from the forward transform, or from the inverse transform, of lengths measured on those axes, should be identical. In other words $\phi(v) = \phi(-v)$. When this scale factor is correctly interpreted as a unit conversion, this identity becomes an explicit statement that both systems employ the same units of measure.

The final conclusion of the paper is that the uniform scale factor must have a value of unity, in order for the identity outlined above, as well as (24), to both hold.

We now have two different assumptions, which allow us to arrive at the canonical, or perhaps we should say restricted, Lorentz Transformation: either we treat the reference frames as being determined by physical particles, which we assume share the same mass, as we discovered in the first section of this paper, or we treat the reference frames as abstract coordinate systems endowed with some unit of measurement, and assume that both systems share the same unit, as was implicitly done in Einstein's 1905 Special Relativity paper.

The equivalent outcome of invoking these two different assumptions draws a vague connection between the mass of an object, and that object's fundamental unit of measurement. We will solidify and expand upon that connection in the next section.

4 Significance of α as a Consequence of Wave Properties

Having stated that we *can* define such a transform, which incorporates a uniform scale α, one must determine if this definition has physical significance.

To make the case for significance, we will begin by remembering that particles exhibit wave properties. In particular, the energy, momentum, and mass of a particle may each be parameterized in terms of these wave properties. Mass, in particular, corresponds to a Compton wavelength λ_m, as well as a period T_m, which is associated with the Zitterbewegung frequency. Mass is energy at rest, so we can think of this wavelength λ_m and period T_m as being associated with a wave which is at rest, or in other words a standing wave.

These standing wave parameters are related to the mass of a particle as follows:

$$T_m = \frac{h}{mc^2} \tag{25}$$

$$\lambda_m = \frac{h}{mc} \tag{26}$$

Therefore the scale factor α may be defined as:

$$\alpha = \frac{m}{m'} = \frac{T'_m}{T_m} = \frac{\lambda'_m}{\lambda_m} \tag{27}$$

If we take two particles, with different masses, to be at rest relative to each other, then the transformation is:

$$t' = \alpha t = \frac{T'_m}{T_m} t \tag{28}$$

$$x' = \alpha x = \frac{\lambda'_m}{\lambda_m} x \tag{29}$$

This simple relationship makes sense if we think about T_m as the fundamental unit of time for a particle, and t being the measure of time with respect to that unit. Likewise, with λ_m as the unit of space, and x as a measure of space with respect to that unit. This being the case, the transformation above is essentially just a unit conversion. The standing waves that correspond to the mass of the particles, end up defining a fundamental unit of measure.

Einstein originally proposed that we think of space and time in terms of tools by which space and time are measured, namely rods and clocks. We discover that if we think of reference frames concretely, in terms of massive particles, and if we think of those particles as exhibiting wave-like properties, then the fundamental rods and clocks reveal themselves as arising from those wave properties.

Wavelengths are rods. Frequencies are clocks.

5 Extended Lorentz Transform in Arbitrary Directions

The canonical (or restricted) Lorentz Transform in an arbitrary direction can be defined as:

$$ct' = \gamma' ct - \gamma' \vec{\beta'} \cdot \vec{x} \tag{30}$$

$$\vec{x}' = \vec{x} + \frac{\gamma' - 1}{\beta'^2} \left(\vec{\beta'} \cdot \vec{x} \right) \vec{\beta'} - \gamma' \vec{\beta'} ct \tag{31}$$

It should be noted that the appearance of $\gamma - 1$ in equation (31) is appropriate for particles, while anti-particles would bear a $\gamma + 1$. That distinction is noted, but not explored in this paper.

The Extended Lorentz Transformation is a space-time rotation accompanied by a space-time scale α, which scale factor is represented as the ratio of the masses of the two physical systems.

$$ct' = \alpha \left(\gamma' ct - \gamma' \vec{\beta'} \cdot \vec{x} \right) \tag{32}$$

$$\vec{x}' = \alpha \left(\vec{x} + \frac{\gamma' - 1}{\beta'^2} \left(\vec{\beta'} \cdot \vec{x} \right) \vec{\beta'} - \gamma' \vec{\beta'} ct \right) \tag{33}$$

If we want to represent the equivalent of equations (13) and (14) where we express the relationship between the space-time coordinates of two particles whose energy and momenta are reported from a common observer, then we have:

$$E't' + \vec{p'} \cdot \vec{x'} = Et + \vec{p} \cdot \vec{x} \tag{34}$$

$$m'c\vec{x'} + \frac{E' - m'c^2}{p'^2 c} \left(\vec{p'} \cdot \vec{x'} \right) \vec{p'} + \vec{p'} ct' = mc\vec{x} + \frac{E - mc^2}{p^2 c} \left(\vec{p} \cdot \vec{x} \right) \vec{p} + \vec{p} ct \tag{35}$$

6 Determining Energy Shifts

One might think that applying a uniform space-time scale α would create results inconsistent with what is measured in reality, particularly for non-unity values of α. For instance, the presence of this uniform scale suggests that if a particle with mass m' emits energy, and then that energy is absorbed by a co-moving particle with a distinctly different mass m, there will be an overall energy shift - a sort of mass based "doppler shift".

On one hand, if we believe that the internal coordinates of particles are determined by their mass, we would absolutely expect measurement with respect to these internal coordinates to be different. Each of these masses internally expresses how many units of length and time are required to make up some kind of standard unit of measurement like a Joule. If space and time are really measured differently for different masses, we would not expect those internal measurements to agree as we go from one mass to the other. On the other hand, if both of the systems have previously agreed on the Joule as a shared *standard* of measurement, and they always report their results in terms of this standard, then the fact that their internal units of measurement differ becomes invisible.

In short, the differences in respective internal measurements are hidden by *calibration*.

The mass m will not observe an energy shift. This is because it will not be comparing the energy it absorbed as seen by m to the energy emitted as seen by m'. Rather it will be comparing the energy absorbed as seen by m to the energy emitted by m' *as seen by an equivalent m that is brought into the same state as m'*. So long as m reports the energy with respect to the calibrated standard of measurement, then m' will stand in perfect agreement.

The only way that a mass based energy shift would ever be observed, would be in the case where the mass of an observer changes. If an observer moving from the state of the absorber into the state of the emitter were to undergo a change in mass by consequence of this move, then we would expect to see a shift in the energy.

The examples which follow include observers acting under the influence of gravitational sources. No doubt, introducing a uniform space-time scale α has a non-trivial affect on geometry, however we make no attempt at this time to directly reconcile the Extended Lorentz Transformation with traditional developments of curved space-time. It should be assumed in what follows that the gravitational fields are weak, and therefore any intermediate, as well as final results should be considered to be approximate. The primary purpose for these examples is to demonstrate the utility, and make a case for the applicability of the Extended Lorentz Transform.

7 Equivalent Masses in a Uniform Gravitational Field

Consider two equal masses m. We place those two masses on two vertically separated shelves within a uniform gravitational field. The masses are at rest relative to each other, so there isn't a relative velocity difference. However, there is a relative mass difference. By moving one of the masses to the higher shelf, a potential energy $m\Delta\Phi$ is added to the rest energy.

If we start off with equations (13) and (14), and take both of the masses to be at rest ($p = 0$; $E = mc^2$), and if we consider that the higher mass has an added $m\Delta\Phi$ worth of rest energy, then we have the relationship:

$$(mc)\,ct' = \left(mc + m\frac{\Delta\Phi}{c}\right)ct \tag{36}$$

$$(mc)\,x' = \left(mc + m\frac{\Delta\Phi}{c}\right)x \tag{37}$$

In other words, an observer moving from the lower shelf to the higher shelf undergoes a change in mass, which introduces a uniform space-time scale:

$$\alpha = 1 + \frac{\Delta\Phi}{c^2} \tag{38}$$

This factor is identical to that achieved by Einstein in his 1911 pre-GR treatment of light moving in a gravitational field. In order to achieve that result, Einstein employed the standard Lorentz Transformation, which is parameterized in terms of relative velocity. However, this tool becomes insufficient if relative velocity is zero. To work around this, Einstein introduced an equivalent moving reference frame. This equivalent frame was accelerated in a manner to specifically replicate the effects of uniform gravitation. The equivalent accelerated frame accrued a change in velocity over the time of flight of the energy, and it is this velocity difference which is used in the Lorentz Transformation to calculate the final result.

Because the Extended Lorentz Transformation is phrased in terms of relative mass, as well as relative velocity, the result may be achieved more directly without needing to introduce an equivalent accelerated reference frame.

However, this value of the uniform scale factor α is merely an approximate value, assuming the field is small, as stated by Einstein himself in the 1911 paper. One way we can convince ourselves of this, is by cutting the distance the observer moves in half, and then taking two consecutive half steps in order to cover the whole distance. For each of these half steps, we accumulate a corresponding scale factor. The scale factor as a product of two half steps differs from the scale factor of a single full step.

We can take this idea to the limit in order to get a more precise representation of the scale factor α. Imagine that the two shelves are separated by a height X. In order to raise the observing object from the lower shelf to the higher shelf, we take N steps.

Let $d\Phi$ correspond to the potential difference due to taking a step $dx = X/N$, as N approaches infinity. The relationship between $d\Phi$ and dx is:

$$d\Phi = dx\frac{\partial\Phi}{\partial x} = \frac{X}{N}\frac{\partial\Phi}{\partial x} \tag{39}$$

At each step, we accumulate a factor $1 + \frac{d\Phi}{c^2}$. The total factor after taking N steps is:

$$\alpha = \prod_i \left(1 + \frac{1}{c^2}\frac{X}{N}\frac{\partial\Phi}{\partial x}\right) \tag{40}$$

If each of these factors were identical, we could merely raise the factor to the power N, and we would be able to recognize the exponent identity. As it turns out, we cannot do this, because the value of each factor is slightly different, due to the derivative of the potential being evaluated at the incremental height corresponding to the factor, rather than a constant position. However, the exponent identity does provide a hint for a path forward.

In order to proceed, we consider that as N becomes large, each factor becomes equivalent to a corresponding exponential factor.

$$1 + \frac{1}{c^2}\frac{X}{N}\frac{\partial \Phi}{\partial x} \approx \exp\left(\frac{1}{c^2}\frac{X}{N}\frac{\partial \Phi}{\partial x}\right) \tag{41}$$

As we plan on letting $N \to \infty$, we allow this approximation to become exact, and we substitute each factor in the product with it's corresponding exponential equivalent.

$$\alpha = \prod_n \exp\left(\frac{1}{c^2}\frac{X}{N}\frac{\partial \Phi}{\partial x}\right) \tag{42}$$

The product of exponents is equal to the exponent of a sum.

$$\alpha = \exp\left(\frac{1}{c^2}\sum_i \frac{X}{N}\frac{\partial \Phi}{\partial x}\right) \tag{43}$$

In the limit $N \to \infty$, the exponential sum becomes an exponential integral.

$$\alpha = \exp\left(\frac{1}{c^2}\int dx \frac{\partial \Phi}{\partial x}\right) = \exp\left(\frac{1}{c^2}\int d\Phi\right) \tag{44}$$

We can now see the correlation between the corrected value of α, and the approximate value.

$$\alpha = \exp\left(\frac{1}{c^2}\int d\Phi\right) \approx 1 + \frac{\Delta\Phi}{c^2} \tag{45}$$

8 Reference Frames of Zero Mass Observers

Because the standard Lorentz Transformation involves an implicit normalization by mc, it cannot be defined for observers with zero mass. However, having removed this normalization, we may now start to examine relationships between the local coordinates of zero mass observers.

To begin with, equations (34) and (35) are perfectly well defined for observers whose mass approaches zero, such as photons. Such zero mass observers additionally manifest a simple relationship between energy E and momentum p.

$$E = pc \tag{46}$$

Inserting this relationship into (34) and (35) collapses four equations into a single equation:

$$E' \left(\vec{n'} \cdot \vec{x'} + ct' \right) = E \left(\vec{n} \cdot \vec{x} + ct \right) \tag{47}$$

Here, the vector $\vec{n}$ represents the direction of motion for each of the massless particles. The primary thing to note, is that this expression cuts out information about some of our spatial directions. For instance, if both zero mass observers were traveling in the same direction, this expression does nothing to relate any measurements orthogonal to the direction of motion.

This can best be understood by returning to the underlying physical mechanism by which these observers measure space and time. Namely, to think of these observing particles as waves, having frequencies as clocks, and wavelengths as rods.

A truly zero mass observer must be represented by a plane wave. Any superposition of plane waves which destroy the planar phase structure of the wave, would also introduce standing wave components, resulting in rest energy, hence mass. Insofar as a wave is approximately planar, it is also approximately mass-less. A purely plane wave may measure time, as it has a frequency, but it may only measure space in one direction - the direction of propagation. A plane wave has no measuring stick (or wavelength) by which it can measure space orthogonal to its own propagating direction.

9 Conclusion

We have discovered that if we attempt to replace the abstract concept of a reference frame, with the more concrete concept of an observing particle, the Lorentz Transform ends up factoring out the mass of the observers. This normalization implies that the standard Lorentz Transform carries an implied assumption that the masses of the particles are identical. As a consequence of removing this assumption, the Lorentz Transformation is extended so that it not only represents a space-time rotation, but also a space-time scale α. The existence of this scale factor α allows us to associate the fundamental wave nature of the particle with the underlying mechanism by which space-time is measured. Namely, it is the wavelength and frequency which take the role of Einstein's rods and clocks. In general, the differences in measure that arise from this space-time scale α can be discounted due to calibration, so long as the scale remains constant. However, in cases where the mass of an observer might change, a non-trivial effect arises. Employing the space-time scale α under such conditions allows us to achieve results which are generally not immediately accessible to Special Relativity. In particular, the influence of gravity may be examined without the need of introducing an equivalent accelerated reference frame. Finally, the removal of normalization allows for realizing relationships between the space-time coordinates of massless observers.

One thing that was intentionally avoided in this paper was an attempt to use this space-time scale factor α in order to understand traditional space-time curvature in the General Relativistic sense. In particular, no attempt was made to describe the origin of the gravitational field, or to speculate causation of that field from the space-time scale α. However, the removal of a core assumption which has been present in Special Relativity for more than a hundred years does indeed invite a review of the development of concepts which depend strongly on SR. One important argument to be made for undertaking such a review is that the identification of the fundamental mechanism by which space-time is perceived, (namely the wave nature of particles), may shed additional light and meaning on what exactly is meant when we talk about space-time, or space-time curvature.