Exploring an Equivalence Interpretation of Cosmic Redshift

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1 Equivalence in Cosmology

At the time when cosmic redshift was first observed, there were two known causes of relativistic redshift, which are connected by an equivalence principle: Doppler shift due to relative motion, and gravitational redshift. Early physicists chose to interpret cosmic redshift initially as a Doppler shift. This subsequently led to our current conception, where cosmic redshift is seen to be a consequence of metric expansion. In this paper we will explore an alternate and equivalent interpretation where redshift is understood in terms of a potential difference, rather than relative motion.

Our understanding of the equivalence principle is formalized by comparing two scenarios. In the first scenario, we have comoving frames (stationary relative to each other) experiencing a difference in gravitational potential. In the second scenario, we have two frames in a vacuum moving at a relative velocity v. These two scenarios are considered equivalent if the potential energy of a test mass in the first scenario equals the kinetic energy of an identical mass in the second scenario. Given this equivalence, the amount of gravitational redshift observed in the first scenario is identical to the amount of Doppler shift observed in an equivalent second scenario.

The velocity v can be used to parameterize the redshift in either scenario. In the first scenario it is taken abstractly, and in the second it is taken literally.

As it pertains to cosmology, these two equivalent scenarios characterize two equivalent modes of redshift. On one hand, redshift described in terms of relative motion without a potential difference, while on the other equivalent hand, redshift is described in terms of a potential difference without relative motion.

With this simple understanding of equivalence, we will derive a potential-induced redshift profile, as a function of distance. This profile has very few parameters to tune, and yet, when an expansion based analysis is applied, we find rough agreement with standard cosmology, in terms of expansion histories, accelerated expansion and dark energy.

2 Modeling the Universe

The model we will use to develop our redshift profile is a universe consisting of an infinite, uniform, and very low gravitating density.

It turns out that this model is anything but simple. The difficulties associated with analyzing an infinite uniform density are well known [3], involving ambiguous boundary conditions and conflicting solutions which are divergent. Care is required in order to properly build, and interpret this model.

To begin, we will presume that the model universe is in a state of equilibrium. Whether that equilibrium is stable or unstable is to be seen. Within this state of equilibrium, we can make a few statements based on the symmetry of the model:

- By symmetry, an observer located at any position will see a balanced matter distribution, so that the gravitational field $\vec{g}$ must be zero at the point of observation.
- Gravity operates to warp the reference frame, affecting how the observer sees the distribution of density at distant locations. By symmetry, an observer at any position cannot claim that the gravitation field $\vec{g}$ is zero everywhere.

The result is that an observer may be considered an inertial observer, only in the local sense. The reference frame of such an observer would be locally inertial, and globally non-inertial. We use the label L_x to refer to the reference frame of a locally inertial observer at x.

There are also signs of trouble, as expected. Due to warping in L_x, an observer at x might determine that there should be a non-zero gravitational field at y, even though, by symmetry, an observer at y detects no such gravitational field. More importantly, if the observer at x doesn't necessarily see a uniform density, due to the warping in the reference frame, then in what sense is the statement of uniform density even meaningful? In which reference frame is the density considered to be uniform?

We will make the statement of uniform density meaningful by introducing a *shared tangent space*. Two locally inertial observers separated by a great distance may have local tangent spaces which are parallel to each other, or in other words, their local tangent spaces are *shared*. The specific hallmark of any two observers who share their local tangent space is that they will both measure the same local value of density.

We construct a globally inertial reference frame G, which is not the reference frame of any single observer, but the shared tangent space of an entire class of observers, spanning all points in the model. Relative to G, we may now make a meaningful statement about a density ρ_g that is globally uniform[1]. The application of symmetry to G indicates that in the shared tangent space, the gravitational field $\vec{g}$ is zero everywhere.

[1]Throughout this paper, the subscript $_g$ will indicate a value that is measured in G, while the corresponding value measured in L_x will be unsubscripted.

Having equipped ourselves with a globally inertial frame G, we now move on to analyze the warped reference frame L_x of a locally inertial observer at point x. Our analysis of L_x will begin in the non-relativistic limit. We have stated that the density ρ_g is very low, so the non-relativistic result should be applicable within an appreciable domain, as well as reveal an intuition about the nature of the problem we are dealing with.

3 L_x in the Classical Limit

The work to construct L_x will be done entirely in the flat and inertial reference frame of G. This is specifically important because only in G does the concept of a uniform density ρ_g make any sense.

The primary difficulty in constructing a local observer frame is determining how gravity warps the coordinates of L_x. In the non-relativistic limit, we can characterize this warping by finding a potential function Φ_x associated with the point x.

We begin by defining a sphere with radius r_g centered on x. This sphere divides the universe into two regions; inside and outside. We can calculate the influence of gravity at points lying on the surface of the sphere by determining the influence of matter from both of those regions.

For the outside region, we imagine that the sphere is surrounded by a shell with thickness h. Due to the shell theorem, we know that points interior to the shell are not gravitationally influenced by any matter inside of h. If we allow $h \to \infty$, we see that the entire outside region of the sphere does not gravitationally influence points on the surface of the sphere at all.

For the inside region, the influence of gravity on surface points is given by a standard radial potential.

$$\Phi_x = -\frac{GM}{r_g} = -\frac{4\pi}{3} G \rho_g r_g^2 \tag{1}$$

This potential has two apparent difficulties, which were previously foreshadowed. The first is that we could have chosen a different point y around which to construct our sphere, which would have led to a radial potential Φ_y whose field gradients conflict with the gradients of Φ_x. The second is that the radial potential diverges, resulting in an eventual infinite force, suggesting a gravitational collapse around the origin point.

We will address those difficulties by interpreting the potential defined in (1) as very specifically describing the potential that the local observer *expects*, and not necessarily as the potential that the local observer sees. This potential warps the coordinates of L_x altering the perception of the local observer. As the potential Φ_x grows large, the warping in the coordinates in L_x becomes more extreme, and therefore more non-inertial.

4 The Fictitious Potential

Making a distinction between *expecting* something and seeing something else may not make sense at first, but remember: L_x is locally inertial and globally non-inertial. Our experience with non-inertial reference frames tells us that fictitious forces will come into play, particularly for large values of r_g. Fictitious forces are artifacts that arise due to non-inertial observers interpreting their own reference frame as if it were inertial. In other words, they *expect* one thing, and end up seeing something else. One way to think of fictitious forces, is that they operate to resolve this conflict. In our case, the conflict can be seen in terms of the description of the field gradients $\vec{g}$, which for G are zero everywhere, and for L_x are not.

To be more specific, the warping of the coordinates in L_x suggest to a local observer that a distant object must fall toward x. However, the inertial coordinates in G predict that such distant objects do not experience any gravitational force at all. The fictitious force which reconciles this conflict is seen by L_x to push radially outward from x, operating as if to keep those distant objects from falling.

Given this interpretation, the ambiguity in the solution of the potential can be recast as an observational degree of freedom. Clearly, by symmetry, any local observer will bear a reference frame that is locally flat, yet globally warped. If the observer at y were to characterize that warping by a potential function Φ_y, we would expect it to be symmetric in form to Φ_x.

More formally, if Φ_x defines the warping in the coordinates of the reference frame L_x, then we would expect the observer at x to observe fictitious forces that would be described by $-\Phi_x$. This fictitious potential not only serves to reconcile the perceived difference in field gradients between L_x and G, but also between L_x and any other observer frame L_y.

In addition to resolving the conflict between alternate reference frames, the fictitious potential also acts to resolve the diverging nature of Φ_x. As the warping in the coordinates diverges, the observed fictitious potential $-\Phi_x$ also diverges in exactly the opposite direction, creating a perfect balance.

Let's now take a moment to summarize.

In an infinite uniform density, we may construct a globally inertial frame G which is the *shared tangent frame* for a class of local observers spanning all points in the universe. Relative to G we may construct the specific reference frame L_x of a local observer positioned at x, which is characterized in the non-relativistic limit by the radial potential Φ_x. For any local observer, this potential takes the same radial form, which implies a locally inertial frame, growing increasingly non-inertial at large radial distance. Because local observers view the universe through a non-inertial frame, artifacts arise at large values of r_g. These artifacts can be described by the apparent action of a fictitious potential $-\Phi_x$, which appears to generate a force that pushes away from x, growing stronger as we move further from x.

5 Calculating Redshift

Returning to our original exercise, we desire to calculate redshift by examining the difference in potential energy of two comoving frames separated by a distance r_g, and equate this to an equivalent kinetic energy. This means we are going to parameterize the difference in potential energy by using an equivalent relative velocity v_g.

According to the results of the previous section, a local detector sees the emitter at a distance r_g as acting under the influence of a fictitious potential $-\Phi_x$. The change in energy U due to this fictitious potential acting on a test mass m is

$$\Delta U = -\frac{4\pi}{3} G m \rho_g r_g^2 \tag{2}$$

The minus sign indicates that the test mass appears to have lost energy as it traversed the distance r_g.

We now equate the magnitude of this potential energy to an equivalent kinetic energy.

$$\frac{1}{2} m v_g^2 = \frac{4\pi}{3} G m \rho_g r_g^2 \tag{3}$$

The value v_g is not the velocity of the emitter. It is the *equivalent* velocity which parameterizes the difference in potential energy.

$$v_g = \left(\frac{8\pi}{3} G \rho_g\right)^{\frac{1}{2}} r_g \tag{4}$$

The test mass m has dropped out of the equation. We see that the velocity v_g is directly proportional to r_g, consistent with observations of cosmic redshift. We can define the constant of proportionality in terms of what we might deem to be the Hubble constant H_0.

$$H_0^2 = \frac{8\pi}{3} G \rho_g \tag{5}$$

This definition of H_0 is perfectly consistent with modern cosmology, in terms of the critical density of a flat Universe. Making this substitution into (4) yields the standard Hubble relation.

$$v_g = H_0 r_g \tag{6}$$

6 The Relativistic Limit

The fact that we have used a classical approach up to this point leads us to conclude that the standard linear Hubble relation is a non-relativistic approximation of a more accurate relativistic redshift law.

If we examine the non-relativistic energy equivalence relation in (3) we can conceive of associated relativistic corrections applied to both the kinetic and potential energies, which will allow us to determine the fully relativistic law.

Kinetic energy can be expressed by subtracting the rest mass from the total energy of a moving test mass.

$$K = \gamma_g mc^2 - mc^2 \tag{7}$$

For potential energy, we will calculate the change in rest mass due to the difference in the fictitious potential. At the emitter, the rest mass is unmodified, but when that mass arrives at the detector it is modified by a factor α_g.

$$U = \alpha_g mc^2 - mc^2 \tag{8}$$

An approximate value of $\alpha_g \approx 1 + \frac{\Delta\Phi}{c^2}$ was determined by Einstein in 1911 [2]. A more correct version of α_g (as calculated in the appendix A) is given by the exponent of the difference of the fictitious potential.

$$\alpha_g = \exp\left(\frac{\Delta\left(-\Phi_x\right)}{c^2}\right) = \exp\left(\frac{H_0^2 r_g^2}{2c^2}\right) \tag{9}$$

If we equate the kinetic and potential energies we get

$$\gamma_g mc^2 - mc^2 = \alpha_g mc^2 - mc^2 \tag{10}$$

The test mass m drops out, and we reduce to an elegant expression of the equivalence principle, where the relativistic factor associated with the energy of motion is equated to the factor associated with potential energy.

$$\gamma_g = \alpha_g \tag{11}$$

This equation is the formal basis for the relativistic Hubble law.

7 Fictitious Metric

We might interpret the equivalence relation in (11) to mean that the correction α_g will have the same effect as γ_g in terms of dilating time, or contracting space. This would be true if α_g represented the warping potential Φ_x, but as it stands, α_g represents the correction due to the *fictitious* potential $-\Phi_x$. This means that the correction is meant to *undo* the warping of the reference frame. In other words, the correction must act to dilate space, and contract time. We use this fact to construct a differential relationship between the coordinates of G and the coordinates of L_x.

$$dt = \frac{1}{\alpha_g} dt_g \tag{12}$$

$$dr = \alpha_g dr_g \tag{13}$$

In order to construct a metric for L_x, we start with the metric for G which is a simple flat spherical coordinate system centered on x. Neglecting rotation, we can assume that the angular differential $d\Omega$ is the same for both global and local observers.

$$ds_g^2 = dt_g^2 - dr_g^2 - r_g^2 d\Omega^2 \tag{14}$$

We can apply the differential relationships (12) to express the metric in terms of dt and dr.

$$ds_g^2 = \alpha_g^2 dt^2 - \frac{dr^2}{\alpha_g^2} - r_g^2 d\Omega^2 \tag{15}$$

One way to describe fictitious forces is that the non-inertial observer neglects the warping of its own coordinates, which we can formalize here by assigning $ds = ds_g$.

The result is called the *fictitious metric*, which we can think of as the observational lens through which local observers experiencing the gravitational influence of the uniform density of the universe.

$$ds^2 = \alpha_g^2 dt^2 - \frac{dr^2}{\alpha_g^2} - r_g^2 d\Omega^2 \tag{16}$$

Note the presence of the global coordinate r_g. While the differential relationship between dr and dr_g is quite simple, determining a value for r_g given a value of r requires numerically solving for the upper limit of the following integral.

$$r = \int_0^{r_g} \exp\left(\frac{H_0^2 x^2}{2c^2}\right) dx \tag{17}$$

Most quantities throughout will be given in terms of r_g, with the expectation that this value can be numerically determined from r when needed.

8 Relativistic Hubble law

Using (11) we can determine a relationship between α_g and v_g, which is essentially the relativistic Hubble law as seen by the abstract global observer.

$$v_g = c\sqrt{1 - \frac{1}{\alpha_g^2}} = c\sqrt{1 - \exp\left(-\frac{H_0^2 r_g^2}{c^2}\right)} \tag{18}$$

To determine the relativistic Hubble law for a local observer, we must first determine the differential relationship between local and global velocity parameters dv and dv_g.

$$dv = d\left(\frac{dr}{dt}\right) = \frac{dr}{dr_g}\frac{dt_g}{dt} d\left(\frac{dr_g}{dt_g}\right) = \alpha_g^2 dv_g \tag{19}$$

We integrate dv over values of r_g (done explicitly in appendix B) in order to discover the locally observed velocity v. The result of this integral can be expressed in terms of the inverse cosh function.

$$v = \frac{c}{2}\cosh^{-1}\left(2\alpha_g^2 - 1\right) = \frac{c}{2}\cosh^{-1}\left(2\exp\left(\frac{H_0^2 r_g^2}{c^2}\right) - 1\right) \tag{20}$$

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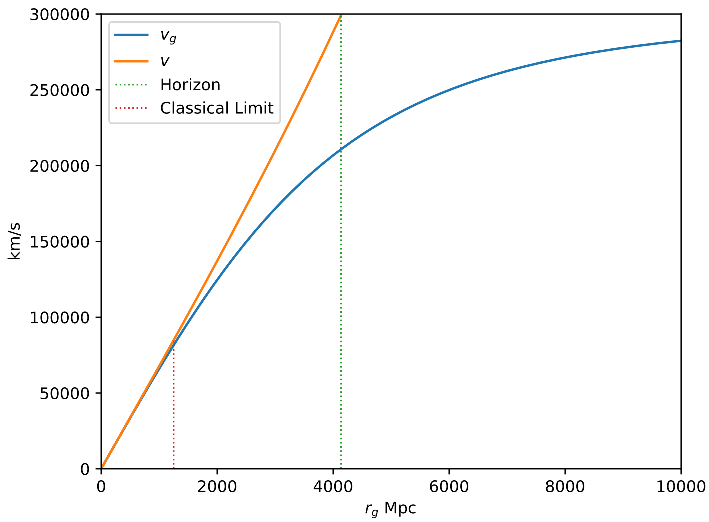

Figure 1: Both the global (18) and local (20) velocity functions approach the linear Hubble law (6) in the non-relativistic limit, which appears to be valid for values of $r_g < 1250$ Mpc. In the relativistic limit, we see a divergence in behavior between the two functions. The velocity v_g seen by the global observer will asymptotically approach the speed of light, while for the local observer the velocity v, having a slightly upward inflection, will cross the speed of light horizon at $r_g = 0.93\frac{c}{H_0}$.

This is the relativistic Hubble law for the local observer, relating the velocity parameter v with the distance r_g from the emitter. This law is defined for all values of r_g (up to the c horizon, and even beyond). The only parameter is the density of the universe, encoded in H_0

Provided the definition (20) for v, we can build an analogy with the energy equivalence relation (11), and define a value for α as seen by the local observer.

$$\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{21}$$

We employ this definition to begin developing useful relationships with the logitudinal redshift parameter z.

$$1+z=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}=\alpha+\sqrt{\alpha^2-1} \tag{22}$$

Inverting this relationship allows us to describe both α and v in terms of redshift.

$$v=c\frac{(1+z)^2-1}{(1+z)^2+1} \tag{23}$$

$$\alpha=\frac{1}{2}\frac{(1+z)^2+1}{1+z} \tag{24}$$

Inverting (20) allows us to define α_g directly in terms of redshift.

$$\alpha_g=\sqrt{\frac{1}{2}\cosh\left(2\frac{(1+z)^2-1}{(1+z)^2+1}\right)+\frac{1}{2}} \tag{25}$$

Inverting (9) allows us to define r_g directly in terms of redshift.

$$r_g=\frac{c}{H_0}\sqrt{\ln\left(\cosh\left(2\frac{(1+z)^2-1}{(1+z)^2+1}\right)+1\right)-\ln(2)} \tag{26}$$

9 Correspondence with Standard Cosmology

The outcome of our analysis, so far, is a well defined Hubble relation, which depends on very few parameters - primarily the value of H_0. This relation describes a redshift, as generated by a fictitious metric, which is the consequence of interpreting observations through a warped coordinate system as if it were not warped.

On the other hand, the model of standard cosmology predicts a redshift relation, as generated by an expanding metric, which depends on many parameters. We have spent the last century fitting those parameters to astronomical observations. Some of these parameters, such as the dark energy parameter, have a well defined value, but an unclear significance.

In order to determine correspondence with standard cosmology, we will apply the same fitting operations to the redshift profile generated by the fictitious metric, and we will see if the parameters - specifically the dark energy parameter - are within a reasonably expected range.

In order to bridge these two interpretations, we recognize that the expansion based interpretation is a function of time, while the non-expansion based model we are currently developing, is a function of space. We use the speed of light to relate the space and time coordinates in the global coordinate system.

$$r_g=-ct_g \tag{27}$$

This expression takes the current moment $t_g = 0$ to be the present, while negative values of time in the past translate to positive radial distances.

In the expansion model, the metric scale parameter a can generally be associated with the redshift. We can also describe the redshift in terms of α in the non-expansion model, as in (22). We can use this fact to develop the correspondence relationship:

$$a = \alpha - \sqrt{\alpha^2 - 1} \tag{28}$$

These relationships establish the basis of correspondence between the expansion and non-expansion models.

10 Friedmann Equations

The FLRW metric of the Big Bang explicitly scales the spatial coordinates uniformly by a time dependent parameter a. The definition of metric expansion allows the Hubble function to be defined in terms of the derivative of a.

$$H = \frac{\dot{a}}{a} \tag{29}$$

Given the correspondence (28), we can determine a Hubble function directly from the redshift of the fictitious metric, labeled H_α.

$$H_\alpha = \alpha^2 H_0^2 \frac{r_g}{v_g} \tag{30}$$

We can equate the Hubble function H_α to the first Friedmann equation in order to determine a radial density function.

$$\rho = \rho_c \alpha^4 H_0^2 \frac{r_g^2}{v_g^2} \tag{31}$$

We can also calculate the derivative of ρ.

$$\dot{\rho} = 2\rho\left(\left(\frac{c}{\alpha_g^2 v_g} - \frac{2\alpha^2 v}{c}\right) H_0^2 \frac{r_g}{v_g} - \frac{c}{r_g}\right) \tag{32}$$

We can insert these values into the continuity equation in order to develop an equation of state, as predicted by the fictitious metric.

$$p = w\rho c^2 \tag{33}$$

$$w = -\left(1 + \frac{2}{3}\left(\frac{c}{\alpha_g^2 \alpha^2 v_g} - \frac{2v}{c} - \frac{v_g c}{\alpha^2 H_0^2 r_g^2}\right)\right) \tag{34}$$

The parameter w can be evaluated as a function of redshift, or time as has been plotted in Fig (2).

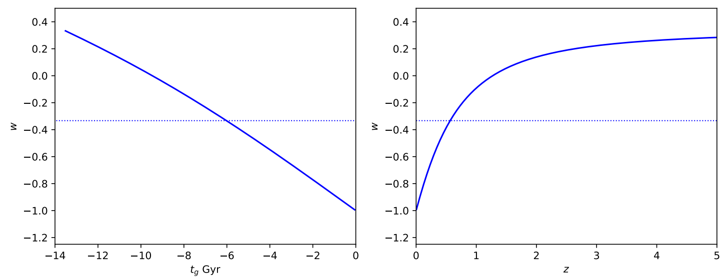

Figure 2: The equation of state w describes how expansion evolves over time. The horizontal dashed line indicates the threshold for accelerated expansion, indicating an epoch of accelerated expansion out to around $z = 0.5$. The current value of w characterizes a universe dominated by dark energy in the form of a cosmological constant, while values corresponding to earlier times characterize a universe dominated by baryonic matter, and even earlier, radiation.

While the overall shape of this curve does not include certain features from standard cosmology such as inflation, there are 3 rough features worth noting. The current value of $w = -1$ characterizes a universe dominated by dark energy in the form of a cosmological constant. The epoch where $w < -1/3$ characterizes a universe in the state of accelerated expansion. The fact that $w \to 1/3$ far in the past characterizes a universe dominated by radiation energy. These epochs roughly align with the picture of standard cosmology.

The redshift calculated from a fictitious metric is entirely consistent with the standard cosmological picture of a universe in a state of accelerated expansion. In the non-expansion model, it is the warping in the coordinates that is 'accelerating'.

11 ΛCDM and Dark Energy

In this section, we will compare the Hubble function H_α determined by the fictitious metric (30) with a corresponding H_Λ based on the ΛCDM model. Our goal will be to fit the parameters of H_Λ to the function H_α, in order to determine consistency with standard cosmology.

In ΛCDM, the density ρ is modeled as being composed of various fractions of energy, based on how those fractions scale with expansion. If we only include the most significant fractions for a flat ($k = 0$) universe, the model reduces to the following simplified expression for H_Λ.

$$H_\Lambda = H_0 \sqrt{\Omega_m \left(1 + z\right)^3 + \Omega_\Lambda} \tag{35}$$

In this simplified model, there are two parameters, but only one degree of freedom. Ω_m represents the fraction attributed to standard matter, and Ω_Λ the fraction attributed to dark energy, and they must sum to unity. The Ω_m term

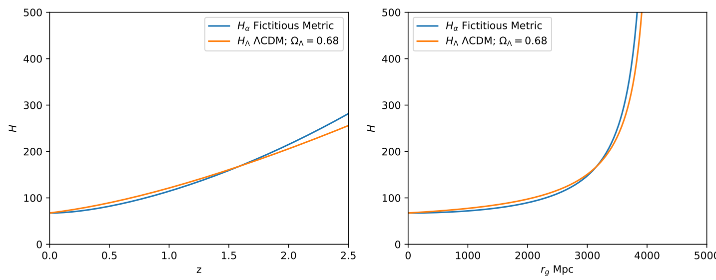

Figure 3: Using the commonly accepted value of $\Omega_\Lambda = 68\%$, we see that H_α intersects with H_Λ at around $z = 1.6$, after which the functions appreciably diverge from each other.

scales as the cube of redshift in order to capture the effect of uniform expansion across all spatial dimensions. The Ω_Λ term is considered to be a constant which does not depend on expansion.

We can optimize the value of Ω_Λ such that the expression for H_Λ aligns as closely as possible with the expression for H_α over some given range.

If the optimization is equally weighted across a range of $z \leq 2.1$, we discover a value for $\Omega_\Lambda = 0.68$. This is the commonly accepted value for dark energy, and the range of optimization is roughly the range of our current astronomical observations for Type Ia supernovae. However, if we extend that range out to $z \leq 2.5$, the optimized value for Ω_Λ drops to 0.65.

A testable prediction of the fictitious metric is that the value of dark energy according to ΛCDM will need to decrease as the data from $z > 2$ begins to dominate the fit for the model.

A A More Correct α

This appendix comes from an earlier paper by the author [1]. The factor α can be expressed in an approximate form in terms of the gravitational potential Φ.

$$\alpha \approx 1 + \frac{\Phi}{c^2} \tag{36}$$

The approximate definition for α is appropriate when the potential is sufficiently small, but we expect this approximation to eventually break down.

In order to create a more accurate value for α we can contemplate moving across the potential difference over many small steps, accumulating factors of the approximate value of α at each step, and then taking the limit of this process.

Let $d\Phi$ correspond to the potential difference due to taking a step dr which is defined in terms of the total distance r divided into N steps, eventually taking N to infinity.

$$d\Phi = dr\frac{\partial \Phi}{\partial r} = \frac{r}{N}\frac{\partial \Phi}{\partial r} \tag{37}$$

For each step, we multiplicatively accumulate a new factor of our approximate version of α.

$$\alpha = \lim_{N\to\infty} \prod_{i}^{N} \left(1 + \frac{r}{N}\frac{1}{c^2}\frac{\partial \Phi_i}{\partial r}\right) \tag{38}$$

Each factor has a slightly different value of $\partial\Phi_i$, but if they were all identical, the product over N factors could be replaced by raising the factor to the N^{th} power, and we would recognize the exponent identity. This exponent identity provides a hint for a path forward.

As N approaches infinity, the factors in our product become equivalent to exponents.

$$\alpha = \lim_{N\to\infty} \prod_{i}^{N} \exp\left(\frac{r}{N}\frac{1}{c^2}\frac{\partial \Phi_i}{\partial r}\right) \tag{39}$$

The product of a factor of exponents is equivalent to the exponent over a sum of terms.

$$\alpha = \lim_{N\to\infty} \exp\left(\sum_{i}^{N} \frac{r}{N}\frac{1}{c^2}\frac{\partial \Phi_i}{\partial r}\right) \tag{40}$$

The sum becomes an integral as we take the limit.

$$\alpha = \exp\left(\frac{1}{c^2}\int dr\frac{\partial \Phi}{\partial r}\right) = \exp\left(\frac{1}{c^2}\int d\Phi\right) = \exp\left(\frac{\Delta\Phi}{c^2}\right) \tag{41}$$

B Integrating dv

The velocity v is expressed as the integral

$$v = \int H_0^2 \frac{r_G}{v_G} dr_G \tag{42}$$

Using an integration variable x, the definite integral can be expressed as

$$\frac{H_0^2}{c}\int_0^{r_G} \frac{x}{\sqrt{1-\exp\left(-\frac{H_0^2 x^2}{c^2}\right)}}dx \tag{43}$$

Multiply the top and bottom by the exponent.

$$\frac{H_0^2}{c}\int_0^{r_G} \frac{x\exp\left(\frac{H_0^2 x^2}{c^2}\right)}{\sqrt{\exp\left(\frac{2H_0^2 x^2}{c^2}\right)-\exp\left(\frac{H_0^2 x^2}{c^2}\right)}}dx \tag{44}$$

Make a u substitution.

$$u = \exp\left(\frac{H_0^2 x^2}{c^2}\right) \tag{45}$$

$$du = 2\frac{H_0^2}{c^2} x \exp\left(\frac{H_0^2 x^2}{c^2}\right) dx \tag{46}$$

Applying the substitution, and resetting the integration variable to x, the integral is now.

$$\frac{c}{2}\int_1^{\alpha_G^2} \frac{dx}{\sqrt{x^2 - x}} \tag{47}$$

We now make another u substitution

$$u = x - \frac{1}{2} \tag{48}$$

$$du = dx \tag{49}$$

$$x = u + \frac{1}{2} \tag{50}$$

$$x^2 = u^2 + u + \frac{1}{4} \tag{51}$$

Applying the substitution, and resetting the integration variable to x, the integral is now.

$$\frac{c}{2}\int_{\frac{1}{2}}^{\alpha_G^2 - \frac{1}{2}} \frac{dx}{\sqrt{x^2 - \frac{1}{4}}} = c\int_{\frac{1}{2}}^{\alpha_G^2 - \frac{1}{2}} \frac{dx}{\sqrt{(2x)^2 - 1}} \tag{52}$$

We now make another u substitution

$$u = 2x \tag{53}$$

$$du = 2dx \tag{54}$$

Applying the substitution, and resetting the integration variable to x, the integral is now.

$$\frac{c}{2}\int_1^{2\alpha_G^2 - 1} \frac{dx}{\sqrt{x^2 - 1}} \tag{55}$$

We recognize the integrand in terms of a hyperbolic trig identity

$$\frac{1}{\sqrt{x^2 - 1}} = \frac{d}{dx}\cosh^{-1}(x) \tag{56}$$

The integral reduces to evaluating the inverse cosh at the end points.

$$v = \frac{c}{2}\left(\cosh^{-1}\left(2\alpha_G^2 - 1\right) - \cosh^{-1}(1)\right) \tag{57}$$

The inverse cosh evaluated at 1 is 0.

$$v = \frac{c}{2}\cosh^{-1}\left(2\alpha_G^2 - 1\right) \tag{58}$$

C Determining H_α

Let the scale factor a be related to the fictitious potential factor α through the observable redshift.

$$a = \frac{1}{1+z} = \alpha - \sqrt{\alpha^2 - 1} \tag{59}$$

The dot operator represents a derivative with respect to the time coordinate t of the local observer.

$$\dot{a} = -a\frac{\dot{\alpha}}{\sqrt{\alpha^2 - 1}} \tag{60}$$

This leads to the expression for H.

$$H = \frac{\dot{a}}{a} = -\frac{\dot{\alpha}}{\sqrt{\alpha^2 - 1}} \tag{61}$$

We can use the definition of α in (21) to evaluate $\dot{\alpha}$.

$$\dot{\alpha} = \frac{v\dot{v}}{c^2\left(1 - \frac{v^2}{c^2}\right)^{3/2}} = \frac{\alpha^3 v}{c^2}\dot{v} \tag{62}$$

Inserting the expression for $\dot{\alpha}$ into (61) gives

$$H = -\frac{\alpha^3 v}{c^2\sqrt{\alpha^2 - 1}}\dot{v} = -\frac{\alpha^2}{c}\dot{v} \tag{63}$$

We can use the definition of v in (20) to evaluate $\dot{v}$.

$$\dot{v} = \frac{c}{2}\frac{d}{dt}\cosh^{-1}\left(2\alpha_g^2 - 1\right) \tag{64}$$

To facilitate this derivative we make a substitution

$$u = 2\alpha_g^2 - 1 \tag{65}$$

$$\dot{v} = \frac{c}{2}\frac{d}{dt}\cosh^{-1}(u) = \frac{c}{2}\frac{du}{dt}\frac{d}{du}\cosh^{-1}(u) \tag{66}$$

We can now employ the inverse hyperbolic trig identity.

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}} \tag{67}$$

This identity holds for $x > 1$, which maps onto the domain of u for values of $\alpha > 1$, or in other words $v > 0$. We make use of this identity to complete the evaluation for $\dot{v}$.

$$\dot{v} = c\frac{2\alpha_g\dot{\alpha_g}}{\sqrt{\left(2\alpha_g^2-1\right)^2-1}} = c\frac{2\alpha_g\dot{\alpha_g}}{\sqrt{4\alpha_g^4-4\alpha_g^2}} = \frac{c}{\sqrt{\alpha_g^2-1}}\dot{\alpha_g} \tag{68}$$

We can proceed with resolving the definition for H.

$$H = -\frac{\alpha^2}{\sqrt{\alpha_g^2-1}}\dot{\alpha_g} \tag{69}$$

We can continue by expressing the derivative of α_g in terms of the derivative of r_g.

$$\dot{\alpha_g} = \alpha_g\frac{H_0^2 r_g}{c^2}\dot{r_g} \tag{70}$$

The next step for our H is

$$H = -\alpha^2 H_0^2\frac{r_g}{cv_g}\dot{r_g} \tag{71}$$

The value $\dot{r_g}$ is the derivative of the radial coordinate in Global Observer space, with respect to Cosmic Time, i.e. the time of the Global Observer. If we think of r_g as representing the position of a photon moving through space, then we can parameterize r_g in terms of t_g. In this case, the derivative is just the speed of light, with an overall minus sign. The minus indicates that as r_g increases, we are talking about times that are further in the past.

$$\dot{r_g} = -c \tag{72}$$

We can now express H in a form that ultimately depends on how cosmic time t_g is defined relative to standard time t.

$$H = \alpha^2 H_0^2\frac{r_g}{v_g} \tag{73}$$

As $r_g \to 0$ we have $v_g \to H_0 r_g$ and $\alpha \to 1$, and therefore $H \to H_0$.

D Adding rotation to the metric

Converting the angular differentials into the local observer's coordinates could potentially be very trivial, with $d\Omega_0 = d\Omega$. However, we can take this opportunity to account for the fact that the local observer could possibly be rotating.

We can express a constant rotation around the polar axis of the spherical coordinate system in terms of a transformation on ϕ_0.

$$\phi = \phi_0 - \omega t_G \tag{74}$$

The corresponding differentials are

$$d\phi = d\phi_0 - \omega \frac{dt_G}{dt} dt \tag{75}$$

Substituting the angular differentials into eq (16) produces

$$ds_G^2 = \left(\alpha'^2 - r_G^2 \sin(\theta) \omega^2 \frac{dt_G}{dt}^2\right) c^2 dt^2 - \frac{dr^2}{\alpha'^2} - r_G^2 d\Omega^2 + \left(2 r_G^2 \sin(\theta) \omega \frac{dt_G}{dt}\right) d\phi dt \tag{76}$$

Isolating the factor scaling the $c^2 dt^2$ term allows us to determine the derivative of time coordinates

$$\frac{dt_G}{dt} = \frac{\alpha'}{\sqrt{1 + r_G^2 \sin(\theta) \omega^2}} \tag{77}$$

Substituting this derivative yields a metric which we call the *cosmic metric*. This is the metric of the global observer, expressed in terms of the reference frame of the local observer. The proposal of this paper is that this metric describes distant observations incorporating the artifacts that arise due to the non-inertial nature of our reference frame.

$$ds_G^2 = \frac{\alpha'^2 c^2 dt^2}{(1 + r_G^2 \sin(\theta) \omega^2)} - \frac{dr^2}{\alpha'^2} - r_G^2 d\Omega^2 + \frac{\left(2\alpha' r_G^2 \sin(\theta) \omega\right) d\phi dt}{\sqrt{1 + r_G^2 \sin(\theta) \omega^2}} \tag{78}$$

We could take this development one step farther, and attempt to incorporate something like centripetal acceleration, but that will not be done in this paper. When concerned with effects that may be due to operating within a rotating reference frame, we can use eq (78) otherwise we can stick with the simpler eq (76), while discussing the cosmic metric.

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